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Morphisms on Semiorthoposets

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A survey concerning morphisms on a semiorthoposet (SOP) is presented. It is pointed out that various types of SOPs possess an order-determining set of morphisms with a specified range. This result is applied to obtain representations of SOPs in terms of SOPs of sets and SOPs of functions. Connections between SOPs and effect algebras as well as tensor products of SOPs are obtained. Proofs of results are omitted and will be presented in a later publication.

1. MOTIVATION

To motivate the structures that we shall subsequently study, we ask the following question. What essential ingredients must a general framework for the foundations of quantum mechanics contain? There appear to be two crucial concepts that quantum mechanics must describe. These two concepts have been given various names, but we shall call them effects and weights. The effects correspond to elementary measurements, observables, events, or experimental propositions for a physical system. These effects usually result from an interaction between the system and a measuring apparatus. Depending on the condition, state, or preparation of the system, each effect carries a certain weight. The weight of an effect gives a measure of the relative likelihood that the effect will occur. Although the weight of an effect is usually taken to be a real number, this need not always be the case. Since we are primarily interested in comparing effects, that is, describing whether one effect is more likely to occur than another, the weight values can be a general set that is capable of ordering likelihoods. A weight is sometimes called a probability measure or a truth function, but we shall allow more general interpretations.

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The set of effects possesses two basic structural properties that we shall call implication and negation. If an effect b occurs whenever an effect a occurs, we say that a *implies* b and write $a \leq b$. Corresponding to an effect a, there is a complementary effect a' called its *negation*. The negation a' occurs if and only if the effect a does not occur. We have not yet assumed any special properties for implication and negation. These properties will follow from two very general, basic axioms that we now present.

Let P and M be the set of effects and weights, respectively, for a physical system. Let $\leq$ be a binary relation on P and let $'$: $P \to P$ be a unary operation. We assume the existence of an *absurd* effect $0 \in P$ that never occurs. Then $1 = 0'$ is the effect that always occurs. A real-valued weight will be described by a $[0, 1]$-*morphism* which is a map m: $P \to [0, 1]$ that satisfies (1) $m(0) = 0$, (2) if $a \leq b$, then $m(a) \leq m(b)$, and (3) $m(a') = 1 - m(a)$ for all $a \in P$. A set of $[0, 1]$-morphisms M is *separating* if $m(a) = m(b)$ for all $m \in M$ implies that $a = b$ and *order determining* if $m(a) \leq m(b)$ for all $m \in M$ implies that $a \leq b$. Our first basic axiom was the existence of $0 \in P$. Our second axiom assumes the existence of a rich supply of weights on P. In the real-valued case this amounts to assuming that there is a separating and order-determining set M of $[0, 1]$-morphisms on P. Physically this means that there are enough experimental conditions or preparation procedures to distinguish between effects and to determine whether one effect implies another. Our first theorem shows that these two axioms imply important structural properties for $\leq$ and $'$.

Theorem 1.1. If P has a separating and order-determining set of $[0, 1]$-morphisms M, then $(P, 0, 1, \leq, ')$ is a bounded poset satisfying:

(i) $a \leq b$ implies $b' \leq a'$	(order inverting)
(ii) $a = a''$ for all $a \in P$	(closed)
(iii) $a \leq a'$ and $b \leq b'$ imply $a \leq b'$	(orthoconsistent)

If we consider morphisms that are not real valued, then we obtain different structures than that given by Theorem 1.1. In these cases, $(P, 0, 1, \leq, ')$ is still a bounded poset satisfying (i). However, (ii) and (iii) are usually replaced by different properties. One of the main results of this article is that the converse of Theorem 1.1 holds. That is, bounded posets satisfying (i)–(iii) are precisely those structures that have a separating and order-determining set of $[0, 1]$-morphisms. Similar results will also be obtained for other types of morphisms.

2. DEFINITIONS

Motivated by the work in Section 1, we now present the general definitions. A *semiorthoposet* (SOP) is a bounded poset $(P, 0, 1, \leq)$ with a unary map $'\colon P \to P$ such that (1) $a \leq b$ implies $b' \leq a'$, and (2) $a \leq a''$ for all $a \in P$. A *unit* SOP is a SOP in which $1' = 0$. An element $a \in P$ is *complementing* if $a \wedge a' = 0$, *sharp* if $a \vee a' = 1$, and *closed* if $a = a''$. If $a \leq b'$, we write $a \perp b$. Notice that if $a \leq b'$, then $b \leq b'' \leq a'$, so $\perp$ is a symmetric relation. An element $a \in P$ is *inconsistent* if $a \perp a$, *strongly inconsistent* if $a = a'$, and *orthoconsistent* if $a \perp a$ and whenever $b \perp b$, then $a \perp b$. Also, a is *consistent* if $a \not\perp a$. The following lemma contains some simple but useful basic results.

Lemma 2.1. A SOP P has the following properties. (i) $a''' = a'$ for all $a \in P$. (ii) $0' = 1$. (iii) If P is a unit SOP and if a is sharp, then a is complementing. (iv) If $a \neq 0$ is complementing, then a is consistent.

The previous definitions give local properties of individual elements. We now give some global definitions. A SOP that is a lattice is called a SOL. A SOP P is *complementing, sharp,* or *closed* if every $a \in P$ is complementing, sharp, or closed, respectively. A SOP P is *consistent* if every $0 \neq a \in P$ is consistent and P is *orthoconsistent* if every inconsistent $a \in P$ is orthoconsistent. It is easy to see that a closed or complementing SOP is a unit SOP. A closed, sharp SOP is called an *orthoposet.*

Lemma 2.2. (i) If P is closed, then $a \in P$ is sharp if and only if a is complementing. (ii) A SOP P is consistent if and only if P is complementing.

If P and Q are SOPs, a *morphism* $\phi\colon P \to Q$ satisfies: $\phi(0) = 0$, $a \leq b$ implies $\phi(a) \leq \phi(b)$, and $\phi(a') = \phi(a)'$ for all $a \in P$. We sometimes call ϕ a *Q-morphism* on P. If, in addition, ϕ is bijective and $\phi(a) \leq \phi(b)$ implies $a \leq b$, then ϕ is an *isomorphism.* A set of Q-morphisms M on P is *order determining* if $m(a) \leq m(b)$ for all $m \in M$ implies $a \leq b$. It is clear that if M is order determining, then M is *separating*; that is, if $m(a) = m(b)$ for all $m \in M$, then $a = b$. A *sub*-SOP P_1 of a SOP P is a subset of P such that $0, 1 \in P_1$ and if $a \in P_1$, then $a' \in P_1$. Notice that a sub-SOP is itself a SOP. A *partial Q-morphism* on P is a Q-morphism defined on a sub-SOP of P.

An *effect algebra* is a system $(P, 0, 1, \oplus)$, where $\oplus$ is a partial binary operation on P satisfying the following:

1. If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
2. If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
3. For every $a \in P$ there exists a unique $a' \in P$ such that $a \oplus a' = 1$.
4. If $a \oplus 1$ is defined, then $a = 0$.

For an effect algebra P, we write $a \perp b$ if $a \oplus b$ is defined. Moreover, we write $a \leq b$ if there exists a $c \in P$ such that $c \perp a$ and $b = a \oplus c$. If

P and Q are effect algebras, an *effect morphism* is a map ϕ: $P \to Q$ such that $\phi(1) = 1$ and $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. An *orthoalgebra* is an effect algebra in which $a \perp a$ implies $a = 0$. The next lemma shows that the notation $a \perp b$ is consistent with our previous usage and that a closed SOP is a generalization of an effect algebra. This result appears in Foulis and Bennett (1994).

Lemma 2.3. (i) If $(P, 0, 1, \oplus)$ is an effect algebra, then $a \perp b$ if and only if $a \leq b'$ and $(P, 0, 1, \leq, ')$ is a closed SOP. (iii) If $(P, 0, 1, \oplus)$ is an orthoalgebra, then $(P, 0, 1, \leq, ')$ is an orthoposet.

Of course, an effect morphism ϕ: $P \to Q$ is a SOP morphism when P and Q are considered as SOPs.

3. EXAMPLES

This section illustrates the generality and unifying power of SOPs by exhibiting a large number of examples.

Example 1. Let X be a nonempty set and let r be a symmetric relation on X. For $A \in 2^X$, define

$$A' = \{y \in X: y\, r\, x \text{ for all } x \in A\}$$

Then $P = \{2^X, \varnothing, X, \subseteq, '\}$ is a SOL. Simple examples show that P need not be a unit SOP and P need not be complementing, sharp, or closed. If r is also irreflexive, then P is complementing, but need not be sharp or closed.

Example 2. Let $L_0(X)$ be the lattice of open subsets of a topological space X. For $A \in L_0(X)$, define $A' = \text{int}(A^c)$. Then $(L_0(X), \varnothing, X, \subseteq, ')$ is a complementing SOL that is not sharp and not closed in general. It is easy to show that $A \in L_0(X)$ is sharp if and only if A is clopen. Moreover, if A is sharp, then A is closed ($A = A''$).

Example 3. Let V be a real or complex inner product space with inner product $\langle x, y\rangle$ and let $L(V)$ be the lattice of all subspaces of V. Define the symmetric relation $x \perp y$ on V by $\langle x, y\rangle = 0$ and for $A \in L(V)$ define

$$A' = \{y \in V: y \perp x \text{ for all } x \in A\}$$

Then $(L(V), \{0\}, V, \subseteq, ')$ is a complementing SOL that is not sharp and not closed, in general. It is easy to show that $A \in L(V)$ is sharp if and only if $A + A' = V$ and that the set of sharp subspaces forms an orthoposet.

Example 4. Let $S = [0, 1] \subseteq \mathbb{R}$ with the usual order $\leq$. For $a \in S$, define $a' = 1 - a$. Then S is a closed SOL that is not complementing (and hence, not sharp). In fact the only complementing (sharp) elements of S are

0 and 1. Moreover, $a \in S$ is consistent if and only if $a > 1/2$. Hence, although S is not consistent, it is orthoconsistent. We can make S into an effect algebra with the same order and $'$ as follows. For $a, b \in S$, we say that $a \oplus b$ exists if $a + b \leq 1$ and we then define $a \oplus b = a + b$.

Example 5. Let X be a nonempty set and let $P = [0, 1]^X$. For $f, g \in P$, define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$ and define $f' = 1 - f$. Then $(P, 0, 1, \leq, ')$ is a closed, orthoconsistent SOL that is not complementing (or sharp) in general. It is easy to show that $f \in P$ is sharp if and only if $f^2 = f$. We can make P into an effect algebra with the same order and $'$ as follows. For $f, g \in P$ we say that $f \oplus g$ is defined if $f(x) + g(x) \leq 1$ for all $x \in X$ and we then define $f \oplus g = f + g$. For $x \in X$, define the effect morphism ϕ_x: $P \to [0, 1]$ by $\phi_x(f) = f(x)$. Then $\{\phi_x: x \in X\}$ is an order-determining set of effect morphisms on P. If P and $[0, 1]$ are considered as SOPs, then $\{\phi_x: x \in X\}$ is an order-determining set of $[0, 1]$-morphisms on P.

Example 6. Let H be a complex Hilbert space and let $E(H)$ be the set of positive linear operators

$$E(H) = \{A: 0 \leq A \leq I\}$$

For $A, B \in E(H)$, we define $A \leq B$ if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$ and we define $A' = I - A$. Then $(E(H), 0, I, \leq, ')$ is a closed, orthoconsistent SOP that is not complementing (or sharp). It is shown in Dvurečenskij (n.d.-b) that $A \in E(H)$ is sharp if and only if $A^2 = A$; that is, A is a projection. We can make $E(H)$ into an effect algebra with the same order and $'$ as follows. For $A, B \in E(H)$ we say that $A \oplus B$ is defined if $A + B \in E(H)$ and we then define $A \oplus B = A + B$. A $[0, 1]$-valued effect morphism on an arbitrary effect algebra is called a *state*. For $x \in H$ with $\|x\| = 1$, define the state ϕ_x on $E(H)$ by $\phi_x(A) = \langle Ax, x\rangle$. Then $\{\phi_x: x \in H\}$ is an order-determining set of states on $E(H)$.

4. FUNCTION SOPs

Let X be a nonempty set and Q be a SOP. For $f, g \in Q^X$ define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$ and define f' by $f'(x) = f(x)'$ for all $x \in X$. Moreover, we define $0, 1 \in Q^X$ by $0(x) = 0$, $1(x) = 1$ for all $x \in X$. Let $P \subseteq Q^X$ satisfy $0 \in P$ and $f \in P$ implies $f' \in P$. Then $(P, 0, 1, \leq, ')$ is a SOP that we call a *Q-function* SOP. The $[0, 1]$-function SOPs are particularly interesting because they correspond to a fuzzy logic or a fuzzy set theory. We shall later show that the class of closed orthoconsistent SOPs coincides with the class of $[0, 1]$-function SOPs.

It is also of interest to consider the sub-SOP $\{0, 1\} \subseteq [0, 1]$. Of course, $\{0, 1\}$ is the simplest nontrivial SOP. Let P be a $\{0, 1\}$-function SOP. If we identify a set with its characteristic function, then P becomes a SOP of sets. Moreover, in this case, if $A \in P$ is considered as a set, then $A' = A^c$. We then call $(P, \emptyset, X, \subseteq, {}^c)$ a *set*-SOP. We shall also show that the class of orthoposets coincides with the class of set-SOPs.

Suppose a SOP P has an order-determining set of Q-morphisms M. Define a map ϕ_M: $P \to Q^M$ by $[\phi_M(a)](m) = m(a)$ and let $(F, P, Q^M) = \{\phi_M(a): a \in P\}$. We frequently denote a Q-function SOP by $(\mathcal{F}, Q^X)$ where $\mathcal{F} \subseteq Q^X$.

Theorem 4.1. (i) If M is an order-determining set of Q-morphisms on P, then (F, P, Q^M) is a Q-function SOP and ϕ_M: $P \to F$ is an isomorphism. (ii) A SOP P has an order-determining set of Q-morphisms if and only if P is isomorphic to a Q-function SOP. (iii) Suppose P has an order determining set of Q-morphisms. If Q is closed, orthoconsistent, complementing, sharp, unit SOP, respectively, then P has these properties, respectively.

Notice that F separates points, because if $m_1, m_2 \in M$ with $m_1 \neq m_2$, then there exists an $a \in P$ such that

$$\phi_M(a)(m_1) = m_1(a) \neq m_2(a) = \phi_M(a)(m_2)$$

The next result shows that (F, P, Q^M) is universal. In this result, M is order determining and is the set of all Q-morphisms on P.

Theorem 4.2. (F, P, Q^M) is universal in the sense that if ϕ: $P \to (\mathcal{F}, Q^X)$ is a morphism, then there exists a unique map α: $X \to M$ such that $\phi(a)(x) = \phi_M(a)(\alpha(x))$ for all $a \in P$, $x \in X$. Moreover, α is injective if and only if $\phi(P)$ separates points.

5. ORDER-DETERMINING MORPHISM SETS

We have seen in Section 4 that a SOP P is isomorphic to a Q-function SOP if and only if P has an order determining set of Q-morphisms. It is thus of interest to know whether P has an order-determining set of Q-morphisms. Moreover, we would like Q to be as simple as possible. For example, we have seen that $Q = \{0, 1\}$ and $Q = [0, 1]$ are important.

We say that a SOP P has *type* $t = 0, 1, 2, 3$, or 4, respectively, if P is an orthoposet, closed and orthoconsistent, closed, sharp and unit, or complementing, respectively. It follows from Theorem 4.1(iii) that if P has an order-determining set of Q-morphisms, then Q must have the same type as P. We thus seek the simplest SOP of each type that is not of a previous type.

If we define $S_0 = \{0, 1\}$ and $S_1 = \{0, 1/2, 1\}$, then S_0 is an orthoposet and S_1 is closed and orthoconsistent but is not an orthoposet. Thus, S_0 has type 0 and S_1 has type 1. The simplest SOP of type 2 that is not of type 1 is $S_2 = \{0, 1, \alpha, \beta\}$, where $\alpha = \alpha' = \alpha''$, $\beta = \beta' = \beta''$, and α, β are unrelated. The simplest SOP of type 3 that is not of type 2 is $S_3 = \{0, 1, \alpha, \alpha', \alpha''\}$, where $0 < \alpha < \alpha'' < 1, 0 < \alpha' < 1$, and there are no other relations. Finally, the simplest SOP of type 4 that is not of type 3 is $S_4 = \{0, 1, \alpha, \omega, \omega'\}$, where

$$0 = \alpha' < \omega' < \alpha < \alpha'' = 1, \qquad 0 < \omega = \omega'' < \alpha < 1$$

and there are no other relations.

The following result shows that any SOP has an abundance of [0, 1]-morphisms, where [0, 1] is the SOP defined in Example 4.

Theorem 5.1. If P is a SOP, then any partial [0, 1]-morphism on P has an extension to a [0, 1]-morphism on P.

To illustrate the utility of the abundance of [0, 1]-morphisms, we now characterize various properties in terms of these morphisms.

Corollary 5.2. Let P be a SOP and let M be the set of all [0, 1]-morphisms on P. (i) $a \in P$ is consistent if and only if there exists an $m \in M$ such that $m(a) = 1$. (ii) $a \in P$ is inconsistent if and only if $m(a) \leq 1/2$ for all $m \in M$. (iii) $a \in P$ is strongly inconsistent if and only if $m(a) = 1/2$ for all $m \in M$. (iv) P is complementing if and only if for every $a \neq 0$ in P there exists an $m \in M$ such that $m(a) = 1$. (v) P is a unit SOP if and only if for every $a \neq 0$ in P there exists an $m \in M$ such that $m(a) \neq 0$.

The next result is a morphism extension theorem for the various types of SOPs.

Theorem 5.3. If P is a SOP of type $t = 0, 1, 2, 3, 4$, then any partial S_t-morphism on P has an extension to an S_t-morphism on P.

This extension theorem can be used to prove the following result.

Theorem 5.4. For a SOP P, the following statements are equivalent. (i) P has type $t = 0, 1, 2, 3$, or 4. (ii) P has an order-determining set of S_t-morphisms. (iii) P is isomorphic to an S_t-function SOP.

Corollary 5.5. For a SOP P, the following statements are equivalent. (i) P is closed and orthoconsistent. (ii) P has an order-determining set of [0, 1]-morphisms. (iii) P is isomorphic to a [0, 1]-function SOP.

That (i) implies (iii) in the next corollary has been proved by Katrnoška (1982).

Corollary 5.6. For a SOP P, the following statements are equivalent. (i) P is an orthoposet. (ii) P has an order-determining set of {0, 1}-morphisms. (iii) P is isomorphic to a set-SOP.

6. EFFECT ALGEBRA IMBEDDINGS

We have seen in Section 2 that an effect algebra is a closed SOP. We now consider an imbedding theorem for closed SOPs into effect algebras.

If P and Q are effect algebras, we define their *horizontal sum* $P \oplus Q$ to be the disjoint union of P and Q with their 0's and 1's identified. For a, $b \in P \oplus Q$ we say that $a \oplus b$ is defined if both a, b are in P and $a \oplus_P b$ is defined or both a, b are in Q and $a \oplus_Q b$ is defined. In the first case we define $a \oplus b = a \oplus_P b$ and in the second $a \oplus b = a \oplus_Q b$. It is easy to check that $P \oplus Q$ is an effect algebra.

Let W be the horizontal sum $W = [0, 1] \oplus [0, 1]$. Then W is an effect algebra and W can also be considered as a closed SOP with the same order and $'$. Since S_2 is a sub-SOP (or sub-effect algebra) of W, we know by Theorem 5.4 that any closed SOP has an order-determining set of W-morphisms. We have introduced W because W-valued effect morphisms are more useful than S_2-valued effect morphisms. The reason for this is that W contains long strings of sums, while S_2 does not. Although a W-valued effect morphisms m is not a state, m can still be interpreted as a probability measure because $m(a) \in [0, 1]$ for any individual a. Moreover, an effect algebra may not have an order-determining set of states, but still may have an order-determining set of W-valued effect morphisms. We now illustrate this with an example.

If H is a Hilbert space, then the effect algebra $E(H)$ of Example 6 in Section 3 has an order-determining set of states. However, the horizontal sum $E(H_1) \oplus E(H_2)$ of two such effect algebras does not have an order-determining (or even separating) set of states. For example,

$$m\left(\frac{1}{n} I_1\right) = \frac{1}{n} = m\left(\frac{1}{n} I_2\right)$$

for every state m. Nevertheless, $E(H_1) \oplus E(H_2)$ has an order-determining set of W-valued effect morphisms. To show this, write W as $W = W_1 \oplus W_2$, W_1, $W_2 = [0, 1]$. If $x \in H_1$, $y \in H_2$, $\|x\| = \|y\| = 1$, define $m_{x,y}$: $E(H_1) \oplus E(H_2) \to W$ by $m_{x,y}(A) = \langle Ax, x\rangle \in W_1$ if $A \in E(H_1)$ and $m_{x,y}(A) = \langle Ay, y\rangle \in W_2$ if $A \in E(H_2)$. It is easy to show that these $m_{x,y}$ form an order-determining set of W-valued effect morphisms on $E(H_1) \oplus E(H_2)$. This result also holds for finite or infinite horizontal sums.

We say that a subset L_1 of an effect algebra L *generates* L if the smallest sub-effect algebra of L that contains L_1 is L itself.

Theorem 6.1. Let P be a closed SOP and let M be an order-determining set of W-morphisms on P. Then there exists an effect algebra Q, an order-determining set of W-valued effect morphisms $\psi(M)$ on Q, a SOP isomorphism ϕ: $P \to \phi(P) \subseteq Q$ such that $\phi(P)$ generates Q, and a bijection ψ: $M \to \psi(M)$

such that $\psi(m)(\phi(a)) = m(a)$ for every $a \in P$, $m \in M$. If Q_1 is an effect algebra with maps ϕ_1: $P \to Q_1$, ψ_1: $M \to \psi_1(M)$ satisfying the previous conditions, then Q_1 and Q are isomorphic. Moreover, if $a, b \in P$ with $a \perp b$ and there exists a $c \in P$ such that $m(c) = m(a) \oplus m(b)$ for all $m \in M$, then $\phi(c) = \phi(a) \oplus \phi(b)$.

Theorem 6.1 gives a canonical imbedding of a closed SOP into an effect algebra that automatically has an order-determining set of W-valued effect morphisms. This result may be important for the axiomatic foundations of quantum mechanics, because the axioms for a closed SOP can be easily justified on physical grounds.

7. BASIC TENSOR PRODUCTS

This section introduces the concept of tensor products of SOPs, presents some examples, and constructs various basic tensor products.

If P, Q, and R are SOPs, a *bimorphism* α: $P \times Q \to R$ satisfies: (1) if $a, b \in P$, $c, d \in Q$, then $a \leq b$ implies that $\alpha(a, c) \leq \alpha(b, c)$ and $c \leq d$ implies that $\alpha(a, c) \leq \alpha(a, d)$; (2) $\alpha(a, c) \perp \alpha(a', d)$ for all $c, d \in Q$ and $\alpha(a, c) \perp \alpha(b, c')$ for all $a, b \in P$; (3) $\alpha(a, 1)' \leq \alpha(a', 1)$ and $\alpha(1, b)' \leq \alpha(1, b')$ for all $a \in P$, $b \in Q$.

Lemma 7.1. Let α: $P \times Q \to R$ be a bimorphism. (i) If $a, b \in P$ and $c, d \in Q$ with $a \leq b$ and $c \leq d$, then $\alpha(a, c) \leq \alpha(b, d)$. (ii) $\alpha(a, 1)' = \alpha(a', 1)$ and $\alpha(1, b)' = \alpha(1, b')$ for all $a \in P$, $b \in Q$. (iii) If $a \perp b$, then $\alpha(a, c) \perp \alpha(b, d)$ for all $c, d \in Q$ and if $c \perp d$, then $\alpha(a, c) \perp \alpha(b, d)$ for all $a, b \in P$. (iv) If R is a unit SOP, then $\alpha(a, 0) = \alpha(0, b) = 0$ for all $a, b \in P$. (v) If R is a unit SOP, then $\alpha(\cdot, 1)$ and $\alpha(1, \cdot)$ are morphisms from P and Q, respectively, into R.

Notice that condition (2) is stronger than the condition $\alpha(a, c) \perp \alpha(a', c)$ that one might expect. Moreover, (2) is stronger than the analogous condition for bimorphisms of orthoalgebras and effect algebras (Dvurečenskij, 1995a; Foulis and Bennett, 1993). We have three replies to this criticism. First, one of the main motivations for studying these structures is to describe physical systems and (2) can be justified on physical grounds. Second, the counterpart to (2) can be assumed in orthoalgebras and effect algebras to obtain stronger results. Third, (2) automatically holds in many natural examples.

Example 1. Define α: $[0, 1] \times [0, 1] \to [0, 1]$ by $\alpha(a, b) = ab$. It is clear that (1) and (3) hold. To prove (2), for any $a, c, d \in [0, 1]$ we have

$$a(c - d) \leq a(1 - d) \leq 1 - d$$

Hence,

$$ac \leq 1 - d + ad = 1 - (1 - a)d = (a'd)'$$

so $\alpha(a, c) \perp \alpha(a', d)$.

Example 2. Let P be a SOL and define α: $P \times P \to P$ by $\alpha(a, b) = a \wedge b$. Again, it is clear that (1) and (3) hold. To prove (2) we have

$$a \wedge c \leq a \leq a'' \leq a'' \vee d' \leq (a' \wedge d)'$$

Hence, $\alpha(a, c) \perp \alpha(a', d)$.

Example 3. Let $P = 2^X$, $Q = 2^Y$, $R = 2^{X\times Y}$ be standard set-SOPs and define α: $P \times Q \to R$ by $\alpha(A, B) = A \times B$, where $A \times B$ is the usual set-theoretic Cartesian product. It is clear that (1)–(3) hold.

Example 4. Let $P = [0, 1]^X$, $Q = [0, 1]^Y$, $R = [0, 1]^{X\times Y}$ be [0, 1]-function SOPs and define α: $P \times Q \to R$ by $\alpha(f, g) = f \otimes g$, where $f \otimes g(x, y) = f(x)g(y)$. Then (1), (3) clearly hold and (2) follows from Example 1.

Example 5. As in Example 6 of Section 3, let $P = \{A \in B(H_1)\colon 0 \leq A \leq I\}$, $Q = \{A \in B(H_2)\colon 0 \leq A \leq I\}$, and $R = \{A \in B(H_1 \otimes H_2)\colon 0 \leq A \leq I\}$, where $H_1 \otimes H_2$ is the usual tensor product and define α: $P \times Q \to R$ by $\alpha(A, B) = A \otimes B$. It is easy to verify (1) and (3). To verify (2), let $A \in P$ and $C, D \in Q$. We must show that

$$A \otimes C \leq I \otimes I - (I - A) \otimes D$$

This is equivalent to showing that

$$\langle Ax, x\rangle\langle Cy, y\rangle \leq \|x\|^2\|y\|^2 - \|x\|^2\langle Dy, y\rangle + \langle Ax, x\rangle\langle Dy, y\rangle$$

for all $x \in H_1$, $y \in H_2$. By Example 1, this inequality holds for all unit vector $x \in H_1$, $y \in H_2$. It then follows that the inequality holds for all $x \in H_1$, $y \in H_2$.

Let P and Q be SOPs of the same type. A *tensor product* of P and Q is a pair (T, τ), where T is a SOP of this type and τ: $P \times Q \to T$ is a bimorphism satisfying: (1) if α: $P \times Q \to R$ is a bimorphism, where R is of this type, then there exists a morphism ϕ: $T \to R$ such that $\alpha = \phi \circ \tau$, (2) T is generated by $\tau(P \times Q)$.

Lemma 7.2. If (T, τ) and (T^*, τ^*) are tensor products of P and Q, then there exists a unique isomorphism ϕ: $T \to T^*$ such that $\tau^* = \phi \circ \tau$.

Lemma 7.2 states that if the tensor product of two SOPs exists, it is unique to within an isomorphism. We now construct some basic tensor products. First, consider the orthoposet $S_0 = \{0, 1\}$.

Lemma 7.3. The tensor product of S_0 and S_0 is (S_0, τ_0), where $\tau_0(a, b) = ab$ for all $a, b \in S_0$.

Next, consider the closed, orthoconsistent SOP $S_1 = \{0, 1/2, 1\}$. Define the SOP

$$S_1 \otimes S_1 = \{0, 1/4, 1/2, 3/4, 1\}$$

where $S_1 \otimes S_1$ is considered as a sub-SOP of $[0, 1]$. Then $S_1 \otimes S_1$ is a closed, orthoconsistent SOP.

Theorem 7.4. The tensor product of S_1 and S_1 is $(S_1 \otimes S_1, \tau_1)$, where $\tau_1(a, b) = ab$ for all $a, b \in S_1$.

We now come to the closed SOP $S_2 = \{0, 1, \alpha, \beta\}$, where $\alpha = \alpha'$, $\beta = \beta'$. Define the set

$$\begin{aligned} S_2 \otimes S_2 = \{0 \otimes 0\} &\cup \{a \otimes b\colon a, b \in S_2, a, b \neq 0\} \\ &\cup \{(a \otimes b)'\colon a, b \in S_2, a, b \neq 0, 1\} \end{aligned}$$

For $\gamma \in S_2 \otimes S_2$ define $0 \otimes 0 \leq \gamma \leq 1 \otimes 1$ and for $a, b \in \{\alpha, \beta\}$ define $a \otimes b \leq a \otimes 1$, $1 \otimes b \leq (a \otimes b)'$. Define $(0 \otimes 0)' = 1 \otimes 1$, $(1 \otimes 1)' = 0 \otimes 0$ and for $a \in \{\alpha, \beta\}$ define $(1 \otimes a)' = 1 \otimes a$, $(a \otimes 1)' = a \otimes 1$. Finally, for $a, b \in \{\alpha, \beta\}$ define the $'$ of $a \otimes b$ as the notation indicates.

Theorem 7.5. The tensor product of S_2 and S_2 is $(S_2 \otimes S_2, \tau_2)$, where $\tau_2(a, 0) = \tau_2(0, a) = 0 \otimes 0$ for all $a \in S_2$ and $\tau_2(a, b) = a \otimes b$ for all $a, b \neq 0$.

Continuing with this program, we have the sharp SOP $S_3 = \{0, 1, \alpha, \alpha', \alpha''\}$. Define the set

$$\begin{aligned} S_3 \otimes S_3 = \{0 \otimes 0\} &\cup \{a \otimes b\colon a, b \in S_3, a, b \neq 0\} \\ &\cup \{(a \otimes b)'\colon a, b \in S_3, a, b \neq 0, 1\} \\ &\cup \{(a \otimes b)''\colon a, b \in S_3, a, b \neq 0, 1\} \end{aligned}$$

Define $(0 \otimes 0)' = 1 \otimes 1$, $(1 \otimes 1)' = 0 \otimes 0$, and for $a \in \{\alpha, \alpha', \alpha''\}$ define $(1 \otimes a)' = 1 \otimes a'$, $(a \otimes 1)' = a' \otimes 1$. Finally, for $a, b \in \{\alpha, \alpha', \alpha''\}$ define the $'$ and the $''$ of $a \otimes b$ as notation indicates. We next define $\leq$ on $S_3 \otimes S_3$ as follows:

(1) $a \otimes b \leq c \otimes d$ if $a \leq c$ and $b \leq d$.
(2) $(a \otimes b)' \leq c \otimes d$ if a or $b = 1$ and this reduces to (1).
(3) $a \otimes b \leq (c \otimes d)'$ if $a \leq c'$ or $b \leq d'$.
(4) $(a \otimes b)' \leq (c \otimes d)'$ if $c \otimes b \leq a \otimes b$.
(5) $(a \otimes b)' \leq (c \otimes d)''$ if $(c \otimes d)' \leq a \otimes b$.

(6) $(a \otimes b)'' \leq (c \otimes d)'$ if $c \otimes d \leq (a \otimes b)'$.
(7) $(a \otimes b)'' \leq (c \otimes d)''$ if $a \otimes b \leq c \otimes d$.
(8) $a \otimes b \leq (c \otimes d)''$ if $a \otimes b \leq c \otimes d$.

Lemma 7.6. The tensor product of S_3 and S_3 is $(S_3 \otimes S_3, \tau_3)$, where $\tau_3(a, 0) = \tau_3(0, a) = 0 \otimes 0$ for all $a \in S_3$ and $\tau_3(a, b) = a \otimes b$ for $a, b \neq 0$.

We also have similar definitions and results for the complementing SOP S_4.

8. TENSOR PRODUCTS OF SOPs

We now give concrete representations of the tensor products of various types of SOPs. This is unlike the orthoalgebra or effect algebra theory, where tensor products are constructed abstractly in terms of the set of all bimorphisms (Dvurečenskij, 1995a; Foulis and Bennett, 1993).

Theorem 8.1. If $P \subseteq Q^X$ and $R \subseteq Q^Y$ are Q-function SOPs, where $Q = S_0, S_1, S_2, S_3$, or S_4, then the tensor product of P and R exists.

Corollary 8.2. Let $P \subseteq Q^X$ and $R \subseteq Q^Y$ be Q-function SOPs where $Q = S_0, S_1, S_2, S_3$, or S_4. Then the tensor product $(P \otimes R, \tau)$ of P and R is a sub-SOP of $(Q \otimes Q)^{X \times Y}$ and $\tau(f, g)(x, y) = f(x) \otimes g(y)$ for all $f \in P$, $g \in R$, $x \in X$, $y \in Y$.

Corollary 8.3. Let P and R be SOPs of type t, $t = 0, 1, 2, 3, 4$. Then the tensor product $(P \otimes R, \tau)$ of P and R is an $S_t \otimes S_t$-function SOP. Moreover, if μ and ν are S_t-morphisms on P and R, respectively, then there exists a unique $S_t \otimes S_t$-morphism λ on $P \otimes R$ such that $\lambda[\tau(a, b)] = \mu(a) \otimes \nu(b)$ for all $a \in P$, $b \in R$.

Corollary 8.4. Let P and R be orthoposets. Then the tensor product $(P \otimes R, \tau)$ of P and R is a standard set-SOP of subsets of a Cartesian product $X \times Y$ and $\tau(a, b) = A \times B$ where $A \subseteq X$, $B \subseteq Y$. Moreover, if μ and ν are S_0-morphisms on P and Q, respectively, then there exists a unique $\lambda \in X \times Y$ such that $\mu(a)\nu(b) = \chi_{A \times B}(\lambda)$ for all $a \in P$, $b \in R$.

Corollary 8.5. Let P and R be closed, orthoconsistent SOPs. Then the tensor product $(P \otimes R, \tau)$ of P and R is a $[0, 1]$-function SOP of functions on a Cartesian product $X \times Y$ and $\tau(a, b)(x, y) = f(x)g(y)$, where $f \in [0, 1]^X$, $g \in [0, 1]^Y$. Moreover, if μ and ν are $[0, 1]$-morphisms on P and Q, respectively, then there exists a unique $(x, y) \in X \times Y$ such that $\mu(a)\nu(b) = \tau(a, b)(x, y)$ for all $a \in P$, $b \in R$.

REFERENCES

Dvurečenskij, A. (1995a). Tensor product of difference posets, *Transactions of the American Mathematical Society*, **347**, 1043–1057.

Dvurečenskij, A. (1995b). Measures and ⊥-decomposable measures on effects of a Hilbert space, to appear.

Foulis, D., and Bennett, M. K. (1993). Tensor products of orthoalgebras, *Order* **10**, 271–282.

Foulis, D., and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics, *Foundations of Physics*, **24**, 1331–1352.

Katrnoška, F. (1982). On the representation of orthocomplemented posets, *Commentationes Mathematicae Universitatis Carolinae*, **23**, 489–498.